Companion Material to

"To Slerp, Or Not To Slerp"

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By Dr. Xin Li

Computer Science Department
Digipen Institute of Technology

Pseudo code: (*Input*: $q_0=[s_0, v_0]$, $q_n=[s_n, v_n]$, n; *Output*: each q_k in the loop)

```
α  = cos⁻¹(dot(q0, qn));
β  = α / n;
u  = ( s0vn – snv0 + v0×vn) / sin(α);
qc = [cos(β),  sin(β)u];
qk = q0;
for (k=1;  k<n;  ++k)
    qk = qc qk;               // quaternion multiplication
```

Listing 1: Power Function

Pseudo code: (*Input*: $q_0=[s_0, v_0]$, $q_n=[s_n, v_n]$, n; *Output*: each q_k in the loop)

```
α  = cos⁻¹(dot(q0, qn));
β  = α / n;
C  = cos(β);
S  = sin(β);
qk = q0;
q̂k = (qn–cos(α)q0) / sin(α);
for (k=1;  k<n;  ++k) {
   qtmp = Cqk + S q̂k ;          // scalar-quaternion product
   q̂k  = C q̂k – Sqk;           // scalar-quaternion product
   qk  = qtmp;
}
```

Listing 2: Tangent Quaternion

Pseudo code: (*Input*: q_0=[s_0, v_0], q_n=[s_n, v_n] , n; *Output*: each q_k in the loop)

```
α   = cos⁻¹(dot(q0, qn));
β   = α / n;
c   = ck = cos(β) + isin(β);
q̂0  = [qn – cos(α)q0] / sin(α);

for (k=1, ck=c;   k<n,   ++k) {
     qk = ck.a*q0 + ck.b* q̂0 ;      // .a and .b are real and imaginary components
     ck = ck*c;                      // complex number multiplication
}
```

Listing 3: Complex Number

Pseudo code: (*Input*: q_0=[s_0, v_0], q_n=[s_n, v_n] , n; *Output*: each q_k in the loop)

```
α    = cos⁻¹(dot(q0, qn));
β    = α / n;
A    = 2cos(β);
q̂0   = (qn – cos(α)q0) / sin(α);
qk-1 = q0;
qk   = cos(β)q0 + sin(β) q̂0 ;

for (k=2;   k<n;   ++k) {
     qtmp = qk;
     qk   = Aqk – qk-1;              // Chebyshev recurrence
     qk-1 = qtmp;
}
```

Listing-4: Chebyshev Sequence

Appendix 3.1

By definition

$$|s_0 v_n - s_n v_0 \ + \ v_0 \times v_n| = \sqrt{(s_0 v_n - s_n v_0 \ + \ v_0 \times v_n)^2}$$

$$= \sqrt{s_0{}^2 v_n{}^2 + s_n{}^2 v_0{}^2 \ + \ (v_0 \times v_n)^2 - 2 s_0 s_n v_0 v_n + s_n v_0 \cdot (v_0 \times v_n) - s_n v_0 \cdot (v_0 \times v_n)}$$

Since $(v_0 \times v_n)^2 =$ (*Lagrange's Identity*), and since $s_n v_0 \cdot (v_0 \times v_n)$ and $s_n v_0 \cdot (v_0 \times v_n)$ are both zeros, we replace, rearrange, regroup and obtain

$$|s_0 v_n - s_n v_0 \ + \ v_0 \times v_n| = \sqrt{s_0{}^2 v_n{}^2 + s_n{}^2 v_0{}^2 \ + \ v_0{}^2 v_n{}^2 - (v_0 \cdot v_n)^2 - 2 s_0 s_n v_0 v_n}$$

$$= \sqrt{s_0{}^2 v_n{}^2 + s_n{}^2 v_0{}^2 \ + \ v_0{}^2 v_n{}^2 + \ s_0{}^2 s_n{}^2 - \ s_0{}^2 s_n{}^2 - 2 s_0 s_n v_0 v_n - (v_0 \cdot v_n)^2}$$

$$= \sqrt{(s_0{}^2 + v_0{}^2)(v_n{}^2 + s_n{}^2) - ((s_0 s_n)^2 + 2 s_0 s_n v_0 v_n + (v_0 \cdot v_n)^2)}$$

$$= \sqrt{1 - (q_0 \cdot q_n)^2}$$

$$= \sqrt{1 - \cos^2(\alpha)} = \sin(\alpha)$$

Appendix 3.2

Let [1, 0] be an identity quaternion.

$$q_k = \left(\sin(\alpha - k\beta)\, q_0 + \sin(k\beta)\, [\cos(\alpha),\ \sin(\alpha)\, u]\, q_0\right) / \sin(\alpha)$$

$$= \left(\sin(\alpha - k\beta) [1, 0]\, q_0 + \sin(k\beta)\, [\cos(\alpha),\ \sin(\alpha)\, u]\, q_0\right) / \sin(\alpha)$$

$$= \left(\sin(\alpha - k\beta) [1, 0] + \sin(k\beta)\, [\cos(\alpha),\ \sin(\alpha)\, u]\,\right) q_0 / \sin(\alpha)$$

$$= \left((\sin(\alpha)\cos(k\beta) - \cos(\alpha)\sin(k\beta)) [1, 0] + \sin(k\beta)\, [\cos(\alpha),\ \sin(\alpha)\, u]\,\right) q_0 / \sin(\alpha)$$

$$= \left([\sin(\alpha)\cos(k\beta),\ \sin(k\beta)\sin(\alpha)\, u]\,\right) q_0 / \sin(\alpha)$$

$$= [\cos(k\beta),\ \sin(k\beta)\, u]\, q_0$$

Appendix 4.1

Let $q_n=[s_n, v_n]$ and $q_0=[s_0, v_0]$ be unit quaternions.

[1] $\hat{q}_k$ is a unit quaternion.

$$
\begin{aligned}
|\hat{q}_k| &= \sqrt{\frac{(\cos(k\beta)s_n-\cos(\alpha-k\beta)s_0)^2}{\sin(\alpha)^2}+\frac{(\cos(k\beta)v_n-\cos(\alpha-k\beta)v_0)^2}{\sin(\alpha)^2}} \\
&= \frac{1}{\sin(\alpha)}\sqrt{\cos(\alpha-k\beta)^2+\cos(k\beta)^2-2\cos(\alpha-k\beta)\cos(k\beta)(q_0\cdot q_n)} \\
&= \frac{1}{\sin(\alpha)}\sqrt{\cos(\alpha-k\beta)^2+\cos(k\beta)^2-2\cos(\alpha-k\beta)\cos(k\beta)\cos(\alpha)} \\
&= \frac{1}{\sin(\alpha)}\sqrt{\sin(\alpha)^2} \\
&= 1
\end{aligned}
$$

[2] The dot product $q_k \cdot \hat{q}_k$ equals zero.

$$
\begin{aligned}
q_k \cdot \hat{q}_k &= \frac{1}{\sin^2(\alpha)}(\sin(k\beta)q_n+\sin(\alpha-k\beta)q_0)\cdot(\cos(k\beta)q_n-\cos(\alpha-k\beta)q_0) \\
&= \frac{1}{\sin^2(\alpha)}(\ \sin(k\beta)\cos(k\beta)+\sin(k\alpha)\cos(\alpha-k\beta)\cos(\alpha) \\
&\qquad +\sin(\alpha-k\beta)\cos(k\beta)\cos(\alpha)-\sin(\alpha-k\beta)\cos(\alpha-k\beta)) \\
&= \frac{1}{\sin^2(\alpha)}(\sin(\alpha-k\beta)(\cos(\alpha)\cos(k\beta)+\sin(\alpha)\sin(k\beta)) \\
&\qquad +\sin(k\beta)\cos(\alpha-k\beta)\cos(\alpha) \\
&\qquad -\sin(\alpha-k\beta)\cos(k\beta)\cos(\alpha)-\sin(k\beta)\cos(k\beta)) \\
&= \frac{1}{\sin^2(\alpha)}(\sin^2(\alpha)\cos(k\beta)\sin(k\beta)+\sin(k\beta)\cos^2(\alpha)\cos(k\beta)-\sin(k\beta)\cos(k\beta)) \\
&= \frac{1}{\sin^2(\alpha)}(\cos(k\beta)\sin(k\beta)(\sin^2(\alpha)+\cos^2(\alpha))-\sin(k\beta)\cos(k\beta)) \\
&= \frac{1}{\sin^2(\alpha)}(\cos(k\beta)\sin(k\beta)-\sin(k\beta)\cos(k\beta)) \\
&= 0
\end{aligned}
$$

Appendix 4.2

$$
\begin{aligned}
q_{k+1} &= \frac{1}{\sin(\alpha)}(\sin((k+1)\beta)q_n + \sin(\alpha-(k+1)\beta)q_0) \\
&= \frac{1}{\sin(\alpha)}(\sin((k\beta+\beta)q_n + \sin(\alpha-k\beta-\beta)q_0) \\
&= \frac{1}{\sin(\alpha)}(\sin(k\beta)\cos(\beta)q_n + \cos(k\beta)\sin(\beta)q_n + \\
&\qquad \sin(\alpha-k\beta)\cos(\beta)q_0 - \cos(\alpha-k\beta)\sin(\beta)q_0) \\
&= \frac{\cos(\beta)}{\sin(\alpha)}(\sin(k\beta)q_n + \sin(\alpha-k\beta)q_0) + \frac{\sin(\beta)}{\sin(\alpha)}(\cos(k\beta)q_n - \cos(\alpha-k\beta)q_0)) \\
&= \cos(\beta)q_k + \sin(\beta)\hat{q}_k
\end{aligned}
$$

$$
\begin{aligned}
\hat{q}_{k+1} &= \frac{1}{\sin(\alpha)}(\cos((k+1)\beta)q_n - \cos(\alpha-(k+1)\beta)q_0) \\
&= \frac{1}{\sin(\alpha)}(\cos(k\beta+\beta)q_n - \cos(\alpha-k\beta-\beta)q_0) \\
&= \frac{1}{\sin(\alpha)}(\cos(k\beta)\cos(\beta)q_n - \sin(k\beta)\sin(\beta)q_n - \\
&\qquad \cos(\alpha-k\beta)\cos(\beta)q_0 - \sin(\alpha-k\beta)\sin(\beta)q_0) \\
&= \frac{\cos(\beta)}{\sin(\alpha)}(\cos(k\beta)q_n - \cos(\alpha-k\beta)q_0) - \frac{\sin(\beta)}{\sin(\alpha)}(\sin(k\beta)q_n + \sin(\alpha-k\beta)q_0) \\
&= \cos(\beta)\hat{q}_k - \sin(\beta)q_k
\end{aligned}
$$

Appendix 6.1

Mathematical Induction:

From (Equation 6.2) we have

$$q_0(\cos(\beta)) = \cos(0)q_0 + \sin(0)\hat{q}_0 = q_0$$

$$q_1(\cos(\beta)) = \cos(\beta)q_0 + \sin(\beta)\hat{q}_0$$

[1] Basis: when k=1

$$\begin{aligned}
q_2(\cos(\beta)) &= \cos(2\beta)q_0 + \sin(2\beta)\hat{q}_{0,} \\
&= \cos^2(\beta)q_0 - \sin^2(\beta)q_0 + 2\sin(\beta)\cos(\beta)\hat{q}_0 \\
&= \cos^2(\beta)q_0 - (1-\cos^2(\beta))q_0 + 2\sin(\beta)\cos(\beta)\hat{q}_0 \\
&= 2\cos^2(\beta)q_0 + 2\sin(\beta)\cos(\beta)\hat{q}_0 - q_0 \\
&= 2\cos(\beta)(\cos(\beta)q_0 + \sin(\beta)\hat{q}_0) - q_0 \\
&= 2\cos(\beta)q_1(\cos(\beta)) - q_0(\cos(\beta))
\end{aligned}$$

[2] Hypothesis: Assume it is true for any k>1

$$q_{k-1}(\cos(\beta)) = \cos((k-1)\beta)q_0 + \sin((k-1)\beta)\hat{q}_0$$

$$q_k(\cos(\beta)) = \cos(k\beta)q_0 + \sin(k\beta)\hat{q}_{0,}$$

[3] Induction: Prove when k+1

$$\begin{aligned}
q_{k+1}(\cos(\beta)) &= \cos((k+1)\beta)q_0 + \sin((k+1)\beta)\hat{q}_0 \\
&= \cos(k\beta)\cos(\beta)q_0 - \sin(k\beta)\sin(\beta)q_0 + \\
&\quad \sin(k\beta)\cos(\beta)\hat{q}_0 + \cos(k\beta)\sin(\beta)\hat{q}_0 \\
&= 2\cos(k\beta)\cos(\beta)q_0 - \cos(k\beta)\cos(\beta)q_0 - \sin(k\beta)\sin(\beta)q_0 + \\
&\quad 2\sin(k\beta)\cos(\beta)\hat{q}_0 - \sin(k\beta)\cos(\beta)\hat{q}_0 + \cos(k\beta)\sin(\beta)\hat{q}_0 \\
&= 2\cos(k\beta)\cos(\beta)q_0 + 2\sin(k\beta)\cos(\beta)\hat{q}_0 - \\
&\quad (\cos(k\beta)\cos(\beta)q_0 + \sin(k\beta)\sin(\beta)q_0 + \\
&\quad \sin(k\beta)\cos(\beta)\hat{q}_0 - \cos(k\beta)\sin(\beta)\hat{q}_0) \\
&= 2\cos(\beta)(\cos(k\beta)q_0 + \sin(k\beta)\hat{q}_0) - (\cos((k-1)\beta)q_0 + \sin((k-1)\beta)\hat{q}_0) \\
&= 2\cos(\beta)q_k(\cos(\beta)) - q_{k-1}(\cos(\beta))
\end{aligned}$$